§7 Noetherian Rings

Recall: A is Noetherian,

i) maximal condition

ii) a.c.c for ideals

iii) $\forall$ ideal is f.g.

Noetherian $\Rightarrow$ Hilbert basis thm & existence of primary decomp.

Prop 7.1 A = noetherian, $A \twoheadrightarrow B$ $\Rightarrow$ B = noetherian

Pf: (6.6).

Prop 7.1. $A \subseteq B$ subring. A = noetherian B = f.g as A-module.

$\Rightarrow$ B = noetherian (as a ring)

Pf: (6.5) $\Rightarrow$ B = noetherian A-module

$\Rightarrow$ B = noetherian B-module

$\Rightarrow$ B = noetherian ring. $\square$

Example: ring of integers in any algebraic number field is noetherian.

Prop 7.3. A = noetherian $\Rightarrow$ $S^{-1}A$ = noetherian.

Pf: (M1) $\{$ideal of $S^{-1}A\}$ $\xleftrightarrow{1:1}$ $\{$contracted ideal of $A\}$

chain / $S^{-1}A$ is stationary $\Leftrightarrow$ chain of contr. ideals is stationary

$\Updownarrow$ $\qquad\qquad\qquad\qquad \Uparrow$

$S^{-1}A$ = noetherian $\qquad\qquad$ A = noetherian

(M2) $\forall\, \mathfrak{b} \triangleleft S^{-1}A \Rightarrow \mathfrak{b} = S^{-1}\mathfrak{a}$

assume $\mathfrak{a} = \sum_{i=1}^{n} A x_i \Rightarrow \mathfrak{b} = \sum_{i=1}^{n} S^{-1}A \cdot \frac{x_i}{1}$ $\square$

Cor 7.4 A = noetherian $\Rightarrow$ $A_{\mathfrak{p}}$ = noetherian.

Thm 7.5 (Hilbert's Basis thm). A = noetherian $\Rightarrow$ $A[x]$ = noetherian

Pf: $\forall\, \mathfrak{a} \triangleleft A[x]$.

$$I := \{\text{leading coeff. of poly. in } \mathfrak{a}\} \triangleleft A$$

$\Rightarrow I \triangleleft A \Rightarrow \mathfrak{a} = (a_1, \cdots, a_n)$

Assume $f_i = a_i x^{r_i} + (\text{lower terms}) \in \mathfrak{a}$.

$r := \max r_i \qquad \mathfrak{a}' := \sum_{i=1}^{n} A[x] f_i \subseteq \mathfrak{a}$

$\forall\, f = a x^m + (\text{lower terms}) \in \mathfrak{a}$

Assume $a = \sum_i u_i a_i$

$\Rightarrow f - \sum_i u_i f_i \cdot x^{m-r_i}$ has degree $< m$

inductively $f = g + h$ where

$\deg g < r$ & $h \in \mathfrak{a}'$

$\Rightarrow \mathfrak{a} = (\mathfrak{a} \cap M) + \mathfrak{a}'$ where $M := \sum_{i=0}^{r-1} A x^i$

M = f.g. A-module

$\Rightarrow \mathfrak{a} \cap M$ = f.g A-module

$$\mathfrak{a} \cap M = \sum_{j=1}^{m} A y_j \qquad y_j \in \mathfrak{a} \cap M$$

$$\Rightarrow \mathfrak{a} = \sum_{j=1}^{m} A y_j + \sum_{i=1}^{n} A[x] f_i$$

$$= \sum_{j=1}^{m} A[x]\, y_j + \sum_{i=1}^{n} A[x] f_i \qquad \square$$

Cor 7.6. A = noetherian $\Rightarrow$ $A[x_1, \dots, x_n]$ = noetherian

Cor 7.7. B = f.g. A-alg. A = noetherian $\Rightarrow$ B = noetherian.

Pf: $B \cong A[x_1 \cdots x_n]/I$ ◻

Example: f.g. ring, f.g. alg over field.

Prop 7.8: $A \subseteq B \subseteq C$ rings.

$$\left.\begin{array}{l} A = \text{noetherian} \\ C = \text{f.g. } A\text{-alg.} \\ C = \text{f.g. } B\text{-mod (or int}/B) \end{array}\right\} \Rightarrow B = \text{f.g. } A\text{-alg.}$$

Pf: $C = A[c_1, c_2, \cdots, c_n]$

$$c_i^{\alpha_i} + b_{i1} c_i^{\alpha_i - 1} + \cdots + b_{i\alpha_i} = 0 \qquad b_{ij} \in B$$

$$B_0 := A[b_{11}, b_{12}, \cdots b_{1\alpha_1}, b_{21}, \cdots, b_{n,\alpha_n}]$$

$$\left.\begin{array}{l} \Rightarrow C = \text{f.g. } B_0\text{-module.} \\ \Rightarrow B_0 = \text{noetherian} \end{array}\right\} \Rightarrow C = \text{noetherian } B_0\text{-module}$$

$\Rightarrow B$ = f.g. B_0-module.

$\Rightarrow B$ = f.g. B_0-alg.

$\Rightarrow B$ = f.g. A-alg.

Prop 7.9 k = field, E = f.g. k-alg.

E = field $\Leftrightarrow$ E = finite alg. ext. of k.

Pf: $E = k[x_1, \cdots, x_n]$.

Suppose E/k not algebraic.

$\Rightarrow$ $x_1, \cdots, x_r$ alg. independent $(r \geq 1)$

& $x_{r+1}, \cdots, x_n$ alg over $k(x_1, \cdots, x_r) =: F$

$$k \subseteq F = k(x_1, x_2, \cdots x_r) \subseteq E$$

Prop 7.8 $\Rightarrow$ F = f.g. k-alg $\Rightarrow$ $F = k[y_1, \cdots, y_r]$

$y_i = \dfrac{f_i}{g_i}$ & $f_i, g_i \in k[x_1 \cdots x_r]$.

$r \neq 0 \Rightarrow \exists$ ∞'ly irr. poly. in $k[x_1 \cdots x_r]$

$\Rightarrow \exists\, h$ irr with $\gcd(h, g_1 \cdots g_r) = 1$.

$\Rightarrow h^{-1} \notin k[y_1 \cdots y_r]$ $\lightning$.

<u>Cor</u> 7.10 (Weak version of Hilbert's Nullstellensatz)

k = field, A = f.g. k-alg. $\mathfrak{m} \triangleleft A$ maximal.

$$\Rightarrow A/\mathfrak{m} = \text{f. alg. ext of } k.$$

In particular, If $k = \bar{k}$, then $A/\mathfrak{m} = k$.

§ 7.2 primary decomposition in noetherian rings.

An ideal $\mathfrak{a}$ is called <u>irreducible</u> if

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \Rightarrow \mathfrak{a} = \mathfrak{b} \text{ or } \mathfrak{a} = \mathfrak{c}$$

Lem 7.11 A = noetherian. every ideal is a finite intersection of irreducible ideals.

Pf: Suppose not. $\mathfrak{a}$ = a maximal element in

$$\Sigma = \{ I \triangleleft A \mid I \text{ is not a f. inters. of irr} \} \neq \phi$$

$\mathfrak{a}$ is not irr. $\Rightarrow \exists\, \mathfrak{b} \,\&\, \mathfrak{c}$ s.t.

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$$

$\mathfrak{a}$ = maximal in $\Sigma \Rightarrow \mathfrak{b}, \mathfrak{c} \notin \Sigma$.

$$\Rightarrow \mathfrak{b} = \bigcap_{i=1}^{n} \mathfrak{b}_i, \quad \mathfrak{c} = \bigcap_{j=1}^{m} \mathfrak{c}_j \qquad \mathfrak{b}_i, \mathfrak{c}_j = \text{irr}$$

$$\Rightarrow \mathfrak{a} = \left(\bigcap_{i=1}^{n} \mathfrak{b}_i \right) \cap \left(\bigcap_{j=1}^{m} \mathfrak{c}_j \right) \qquad \lightning .$$

Lemma 7.12 In a Noetherian ring.

irreducible $\Rightarrow$ primary

Pf: $\mathfrak{a}$ = irr

$\mathfrak{a}$ = primary $\Leftrightarrow$ $\bar{0}$ = primary in $A/\mathfrak{a}$.

WMA : $\mathfrak{a} = 0$.

$\forall\, xy = 0$ with $y \neq 0$.

$$\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x^2) \subseteq \cdots$$

a.c.c $\Rightarrow$ $\mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1})$

$\Rightarrow$ $(x^n) \cap (y) = 0$

$$\left(\begin{array}{l} a \in (x^n) \cap (y) \Rightarrow a = x^n a_0 = y b_0 \\ xy = 0 \Rightarrow x^{n+1} a_0 = xa = xyb_0 = 0 \\ \qquad \Rightarrow x^n a_0 = 0 \Rightarrow a = 0 \end{array} \right)$$

$$\left. \begin{array}{l} (0) = \text{irr} \\ y \neq 0 \end{array} \right\} \Rightarrow (x^n) = 0 \Rightarrow x^n = 0$$

Thm 7.13. Every ideal has primary decomposition in a Noetherian ring.

Prop 7.14. $\mathfrak{a} \triangleleft A$ noetherian. $\exists\, n$ s.t.

$$(\sqrt{\mathfrak{a}})^n \subseteq \mathfrak{a}$$

Pf: $\sqrt{\mathfrak{a}} = \sum_{i=1}^{k} A x_i$. $\quad x_i^{n_i} \in \mathfrak{a} \qquad m = \sum_{i=1}^{k} n_i$

$$\Rightarrow (\sqrt{\mathfrak{a}})^m = \left\{ \sum_{\sum_{i=1}^{k} r_i = m} A\, x_1^{r_1} \cdots x_k^{r_k} \right\} \subseteq \mathfrak{a}$$

$$\left(\begin{aligned} \sum_{i=1}^{k} r_i = m = \sum_{i=1}^{k} n_i \quad &\Rightarrow \exists\, i_0 \text{ s.t. } r_{i_0} \geq n_{i_0} \\ &\Rightarrow x_{i_0}^{r_{i_0}} \in \mathfrak{a} \\ &\Rightarrow x_1^{r_1} \cdots x_k^{r_k} \in \mathfrak{a}. \end{aligned} \right)$$

$\square$

Cor 7.15. In a Noetherian ring, the nilradical is nilpotent.

Pf: $\mathfrak{a} = (0)$. $\square$

Cor 7.16. A = noetherian. $m \triangleleft A$ maximal, $q \triangleleft A$. TFAE:

i) $q = m$-primary

ii) $\sqrt{q} = m$

iii) $m^n \subseteq q \subseteq m$ for some $n > 0$.

Pf: i) $\Rightarrow$ ii) ✓

ii) $\Rightarrow$ i) (4.2)

ii) $\Rightarrow$ iii) (7.14)

iii) $\Rightarrow$ ii) ✓

Prop 7.17. A = noetherian. $\mathfrak{a} \triangleleft A$

$$\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \text{ belongs to } \mathfrak{a}\} = \{(\mathfrak{a} : x) \mid x \in A\} \cap \operatorname{Spec} A$$

Pf: WMA: $\mathfrak{a} = 0$. & $0 = \bigcap_{i=1}^{n} \mathfrak{q}_i$ minimal primary decomp.

$\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$

$\forall x \in \mathfrak{a}_i := \bigcap_{j \neq i} \mathfrak{q}_j \neq 0.$

$$\sqrt{\operatorname{Ann}(x)} = \sqrt{(0 : x)} = \sqrt{(\cap \mathfrak{q}_i : x)}$$

$$= \sqrt{\bigcap_i (\mathfrak{q}_i : x)} = \bigcap_i \sqrt{(\mathfrak{q}_i : x)} = \mathfrak{p}_i$$

$\Rightarrow \operatorname{Ann}(x) \subseteq \mathfrak{p}_i$

(7.14) $\Rightarrow \mathfrak{p}_i^m \subseteq \mathfrak{q}_i \Rightarrow \mathfrak{a}_i \mathfrak{p}_i^m \subseteq \mathfrak{a}_i \cap \mathfrak{p}_i^m \subseteq \mathfrak{a}_i \cap \mathfrak{q}_i = 0$

$m \geq 1$ be the smallest integer s.t. $\mathfrak{a}_i \mathfrak{p}_i^m = 0$.

$$x \in \mathfrak{a}_i \mathfrak{p}_i^{m-1} \setminus \{0\}.$$

$$\Rightarrow \mathfrak{p}_i x = 0 \Rightarrow \operatorname{Ann}(x) \supseteq \mathfrak{p}_i$$

$$\Rightarrow \operatorname{Ann}(x) = \mathfrak{p}_i$$

Conversely, $\mathfrak{p} = \operatorname{Ann}(x) =$ prime ideal

$$\Rightarrow \sqrt{\operatorname{Ann}(x)} = \mathfrak{p} = \text{prime}$$

$$\overset{(4.5)}{\Rightarrow} \mathfrak{p} \text{ belongs to } 0.$$